PROCEEDINGS A

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Supplementary material for: Self-similar slip instability on interfaces with rate- and state-dependent friction

Robert C. Viesca [1]

[1] Tufts University, Medford, MA 02155

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1. Methods for numerical solution for slip rate and state evolution

Here we outline the techniques used to numerically solve for the evolution of the state variable and slip rate on a fault for given initial and boundary conditions. We slightly reformulate the problem: first, the variables to be solved for and, second, the independent variable over which the solution is said to evolve. The evolution of slip rate and state is coupled not only in the sense that the slip rate affects the state evolution and vice-versa at a single point on the fault, but also in the sense that points in space are coupled via the elastic response of the surrounding medium. How this elastic interaction is best evaluated numerically varies according to the form that interaction takes. We focus in the subsequent sub-sections on the methods used to evaluate this term for the configurations considered in the main text.

We replace θ by $\Theta = \ln(V_i\theta/D_c)$, such that (2.1) becomes $f = f_o + a\ln(V/V_i) + \Theta$, and the aging law, (2.3), becomes

$$\frac{\partial\Theta(x,t)}{\partial t} = \frac{V_i}{D_c}\left[\exp\left[-\Theta(x,t)\right] - \frac{V(x,t)}{V_i}\right] \tag{1.1}$$

Requiring $\sigma\partial f/\partial t = \partial\tau/\partial t$ is then equivalent to

$$a\frac{\partial V(x,t)/\partial t}{V(x,t)} + b\frac{\partial\Theta(x,t)}{\partial t} = \frac{1}{\sigma}\left[\frac{\partial\tau_{el}(x,t)}{\partial t} + \frac{\partial\tau_o(x,t)}{\partial t}\right] \tag{1.2}$$

Using (1.1) and (1.2), we may write the rate of V explicitly in terms of V and Θ, and a non-autonomous forcing term $(\partial\tau_o/\partial t)$. This, along with (1.1), forms the system of equations that may be solved at positions along the fault. The natural scaling here is to scale stress by σb, slip velocity by V_i, and time by D_c/V_i, leaving the sole parameter a/b.

We discretize V and Θ at positions x_n $(n = -N, -N+1, ..., N)$ along the fault and denote their values as $V_n(t)$ and $\Theta_n(t)$. We may then arrive to expressions for dV_n/dt and $d\Theta_n/dt$ that are likewise functions of V_n, Θ_n, and the value of $\partial\tau_o/\partial t$ at x_n, as well as the discrete distribution of V along the fault necessary to numerically evaluate the contribution of $\partial\tau_{el}/\partial t$ there as well. In the demonstrative simulations done here, we take the fault to initially be at steady state ($\Phi = 0$) with a uniform sliding velocity V_i. We nucleate the slip instability by imposing a compact rate of external loading $\partial\tau_o/\partial t$, held constant in time beginning at $t = 0^+$ with the spatial distribution

$$\frac{\partial\tau_o(x,t)}{\partial t} = \frac{\sigma b}{D_c/V_i}\left[1 - \left(\frac{x}{L_\tau}\right)^2\right]^{3/2} \tag{1.3}$$

for $|x| \le L_\tau$ and 0 otherwise, where L_τ is the half-width of the loaded region and the specific values chosen given in the caption of Figures with simulation results (for the slip between elastic half-spaces, this has form $L_\tau = L_b/(1-a/b)$ or $L_b/(1-a/b)^2$).

Here, we expect the velocity to diverge as a finite time instability, implying that to resolve a given increment in velocity requires progressively smaller time increments as $t_f \to 0$. To circumvent this, we change the independent variable of V_n and Θ_n from time to the slip at the origin, $\delta_0 = \delta(0,t)$. The convenience in doing so is that, for a coordinate system oriented such that the slip instability develops about the origin (as is done here) and for a strictly positive slip rate that ultimately diverges in the manner of (3.4), δ_0 will have a monotonic relation with t that itself ultimately diverges (logarithmically) while $t_f \to 0$ (such that logarithmic increments of V occur over approximately fixed increments of δ_0). Using the notation $(\cdot)_{n,\delta_0}(\delta_0) = (\cdot)_n[t(\delta_0)]$, the change of variable implies that $dV_{n,\delta_0}/d\delta_0 = (dV_n/dt)(dt/d\delta_0)$ (and likewise for $d\Theta_{n,\delta_0}/d\delta_0$), with $dt(\delta_0)/d\delta_0 = 1/V[0,t(\delta_0)]$. Thus it is the evolution of the pair V_{n,δ_0} and Θ_{n,δ_0} that we solve for at each discrete point by integrating the resulting system ODEs forward in δ_0, as well as solving for $t(\delta_0)$, defining $\delta_0 = 0$ such that $t(0) = 0$. We perform this method-of-lines integration using the adaptive-step integration routine ode113 in MATLAB.

In solving for the evolution of V and Θ, we must numerically evaluate the operator $\mathcal{L}$ associated with $\partial\tau_{el}/\partial t$ at each step. We recall that the form that that operator takes depends on the mode of slip (e.g., in- or anti-plane, or mixed-mode slip) as well as the elastic configuration of the surrounding medium (e.g., material heterogeneities, including the presence of a free surface). In what follows we examine the two procedures used to arrive at results presented in the main text for the various configurations considered.

(a) Fourier-transform approach for in- or anti-plane rupture

Here we outline a commonly employed spectral method that makes use of the Fourier transform [e.g., *King*, 2009] properties of the operator $\mathcal{L}$. We use this technique in finding the solutions presented in Figures 6 and 9. While the method is expedient, it is disadvantageous for precise comparisons with the fixed-point solutions. In the subsequent sub-section, we outline the implementation of an alternative method to alleviate this deficiency.

For single-mode slip at the interface of two elastic half-spaces, $\partial\tau_{el}/\partial t$ is

$$\mathcal{L}(V) = \frac{\bar{\mu}}{2\pi}\int_{-\infty}^{\infty}\frac{\partial V/\partial s}{s-x}ds \tag{1.4}$$

We evaluate the above by evaluating its Fourier transform, which we denote by the operator $\mathcal{F}[f(x)]$ when operating on a function $f(x)$, followed by its inverse, $\mathcal{F}^{-1}$; the transform implicitly returns a function of the wavenumber k. We use the properties of the Fourier transform of convolution and derivatives (here, the spatial derivative of V) to concisely write $\mathcal{F}(\mathcal{L})$ as

$$\mathcal{F}\left[\frac{\partial\tau_{el}(x,t)}{\partial t}\right] = -\mu' ik\mathcal{F}[V(x,t)]\mathcal{F}\left[1/x\right] \tag{1.5}$$

where $\mathcal{F}[1/x] = -\pi i\,\mathrm{sgn}(k_w)$.

Thus, after finding the Fourier transform of a distribution of V at a fixed time, (1.5) can be calculated, and taking its inverse Fourier transform provides the concomitant elastic stressing rate. Numerically, the transform of V and the inverse of (1.5) may be efficiently calculated using fast Fourier transform techniques for V defined as a periodic function on an interval with period $2L_p$ and defined on an evenly spaced grid of $2N-1$ points, $x_n = -L_p,\ -L_p+\Delta x,\ \ldots,\ L_p-\Delta x$.

The periodicity imposed on V introduces an artifact from the elastic interactions among the periodic array when the elasticity is non-local, like for slip between elastic half-spaces. This artifact would diminish when L_p is taken to be increasingly greater than an expected lengthscale for velocity development, here L. The order of the rate of this decrease can be estimated by the decay of the stress rate with distance from the developing slip instability, which for in- or anti-plane rupture would occur as the squared inverse of distance from the instability patch. We typically choose L_p such that $(L/L_p)^2 < 1\%$ for the numerical solutions where this method is implemented.

For slip of a layer of thickness h on an elastically similar substrate, $\partial\tau_{el}/\partial t$ is

$$\mathcal{L}(V) = \bar{E}h\frac{\partial^2 V}{\partial x^2} \tag{1.6}$$

This may also be evaluated by the Fourier transform technique above, and we do so out of convenience in arriving to the solutions in Figure 9. Given the local nature of the interactions embodied by (1.6), these solutions do not suffer from the implicit replication of the domain.

However, a more typical route to numerically evaluating (1.6) would be to use traditional schemes for differential operators (e.g., finite differences) acting over a finite or periodic domain.

(b) Weideman eigenfunction expansion for in- or anti-plane rupture between two half-spaces

As noted above, a deficiency of the Fourier transform approach is the implicit periodic replication of a finite domain. If we are to precisely compare the evolution of a single slip instability (and not a periodic array of them) to the fixed-point solutions of the main text, we require a numerical method that can track evolution of slip rate and state on the entire fault. This is necessary for an independent and accurate evaluation of the asymptotic stability of the fixed points. For the problem of in- or anti-plane rupture of two half-spaces, this requires the Hilbert transform and spatial derivatives to be evaluated numerically on the entire real line.

To accomplish this, we look towards a set of functions that form a complete orthonormal basis on the real line [*Higgins*, 1977; *Wiener*, 1949],

$$\rho_n(x) = \frac{(1+ix)^n}{(1-ix)^{n+1}} \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

whose inner product

$$\int_{-\infty}^{\infty} \rho_n(x)\overline{\rho_m(x)}dx = \pi\delta^K_{n,m}$$

where the overline denotes the complex conjugate, and $\delta^K_{n,m}$ is the Kronecker delta ($\delta^K_{n,m} = 1$ when $n = m$, otherwise $\delta^K_{n,m} = 0$).

This choice of this basis set is motivated by the results of *Christov* [1982], *James and Weideman* [1992], and *Weideman* [1992, 1995] who found an efficient manner for the numerical evaluation of the Hilbert transform and spatial derivatives on the real line and whose key developments we summarize below.

We can represent a real-valued function defined on the real line, $f(x)$, as

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \rho_n(x) \tag{1.7}$$

where the coefficients

$$a_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\overline{\rho_n(x)}dx \tag{1.8}$$

One advantage of using this basis is that ρ_n are eigenfunctions of the Hilbert transform $\mathcal{H}$ [*Weideman*, 1995]

$$\mathcal{H}[\rho_n(x)] = \frac{1}{\pi}\int_{\infty}^{\infty} \frac{\rho_n(s)}{s-x}ds = \lambda_n \rho_n(x) \tag{1.9}$$

with the eigenvalues $\lambda_n = -i\mathrm{sgn}(n)$, where $\mathrm{sgn}(n) = n/|n|$ for $n \neq 0$ and $\mathrm{sgn}(0) = 1$.

Using the coordinate map

$$e^{i\theta} = \frac{1+ix}{1-ix}, \qquad x = \tan\frac{\theta}{2} \tag{1.10}$$

(1.7) can be rewritten as

$$f(x)(1-ix) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

and (1.8) may be reexpressed as

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 - i\tan\frac{\theta}{2}\right) f\left(\tan\frac{\theta}{2}\right) e^{-in\theta}d\theta \tag{1.11}$$

We may truncate the expansion (1.7)

$$f(x)(1-ix) \approx \sum_{n=-N}^{-N+1} a_n e^{in\theta} \tag{1.12}$$

and approximate the solution for the coefficients a_n by discretizing (1.11) via the trapezoidal rule

$$a_n \approx \tilde{a}_n = \frac{1}{2N} \sum_{j=-N}^{N-1} \left(1 - i\tan\frac{\theta_j}{2}\right) f\left(\tan\frac{\theta_j}{2}\right) e^{-in\theta_j} \tag{1.13}$$

for an even grid spacing of $\theta_j = \pi j/N$. The corresponding grid on x, $x_j = \tan(\theta_j/2)$, is unevenly spaced with increased spacing under increasing $|x_j|$. (1.13) takes the form of a discrete Fourier transform, such that the coefficients $\tilde{a}_n$ may be found via the fast Fourier transform (FFT) of a vector consisting of the product of the two terms with parentheses. Once the approximate coefficients $\tilde{a}_n$ are found, we may proceed to approximate both the Hilbert transform and the spatial derivative of f at x_j.

Given the property (1.9), the truncated expansion (1.12), and the approximate expansion coefficients $\tilde{a}_n$, the Hilbert transform of f is approximated as

$$\mathcal{H}[f(x)] \approx \sum_{n=-N}^{N-1} \lambda_n \tilde{a}_n \rho_n(x)$$

Performing the variable transformation (1.10) and considering the points x_j, this may be reexpressed as

$$\mathcal{H}[f(x_j)] \approx \frac{1}{1 - i\tan\frac{\theta_j}{2}} \sum_{n=-N}^{N-1} \lambda_n \tilde{a}_n e^{in\theta_j} \tag{1.14}$$

where the sum has the form of a discrete inverse Fourier transform, such that the Hilbert transform of f at x_j may be found by taking the inverse FFT of the product $\lambda_n \tilde{a}_n$ and dividing the result by $(1 - ix_j)$.

The first derivative of the truncated expansion of (1.7) is simply

$$\frac{df(x)}{dx} \approx \sum_{n=-N}^{N-1} a_n \frac{d\rho_n(x)}{dx} \tag{1.15}$$

However, recognizing that

$$\frac{d\rho_n(x)}{dx} = \frac{i}{2}\left[n\rho_{n-1}(x) + (2n+1)\rho_n(x) + (n+1)\rho_{n+1}(x)\right]$$

we may rewrite (1.15) as

$$\frac{df(x)}{dx} \approx \sum_{n=-N}^{N-1} a_n^{(1)} \rho_n(x)$$

where

$$a_n^{(1)} = \frac{i}{2}\left[n a_{n-1}(x) + (2n+1)a_n(x) + (n+1)a_{n+1}(x)\right] \tag{1.16}$$

such that, after performing the change of variable from x to θ, we find that we may approximate the derivative at x_j by dividing the inverse discrete Fourier transform of $a_n^{(1)}$ by $1 - ix_j$:

$$\frac{df(x_j)}{dx} \approx \frac{1}{1 - i\tan\frac{\theta_j}{2}} \sum_{n=-N}^{N-1} a_n^{(1)} e^{in\theta_j} \tag{1.17}$$

For the in-plane or anti-plane rupture, the functional $\mathcal{L}$ involves the sequence of operations $\mathcal{H}[df(x)/dx]$, whose evaluation at points x_j is a simple extension of the above

$$\mathcal{H}[df(x_j)/dx] \approx \frac{1}{1 - i\tan\dfrac{\theta_j}{2}} \sum_{n=-N}^{N-1} \lambda_n a_n^{(1)} e^{in\theta_j} \tag{1.18}$$

In evaluating $a_n^{(1)}$ in (1.16–1.18), we use the approximated coefficients $\tilde{a}_n$ of (1.13).

In our implementation, we follow *James and Weideman* [1992] and use a slightly different definition of the basis functions ρ_n

$$\rho_n^B(x) = \frac{(B+ix)^n}{(B-ix)^{n+1}} \tag{1.19}$$

with the coordinate transformation instead being

$$e^{i\theta} = \frac{B+ix}{B-ix}, \quad x = \tan\frac{\theta}{2} \quad x = B\tan\frac{\theta}{2} \tag{1.20}$$

The purpose of this change is that the length B is a parameter allowing for the adjustment of the grid x_j to resolve regions undergoing the sharpest change spatially (i.e., within L). Using (1.19) and (1.20) we can rederive the preceding results of this sub-section. The key results (1.13), (1.14), and (1.17), remain the same, except wherever $1 - ix$ appears, it is replaced by $B - ix$, and a factor of $1/B$ is introduced on the right hand side of (1.16).

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